

# ON THE MODIFIED SELBERG INTEGRAL

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**Abstract.** We give a kind of “approximate majorant principle” result for the “modified Selberg integral”, say  $\tilde{J}_f(N, h)$ , of essentially bounded  $f : \mathbb{N} \rightarrow \mathbb{R}$  (i.e., bounded by arbitrary small powers); i.e., we get an upper bound, in terms of the modified Selberg integral of a related function  $F$  (with  $|f * \mu| \ll F * \mu$ , in the supports intersection), getting a “square-root cancellation” for the error-terms. Here  $\tilde{J}_f(N, h)$  is the mean-square (in  $N < x \leq 2N$ ) of the “averaged short sum” of, say,  $f := g * \mathbf{1}$ , minus its expected value; i.e.,  $\frac{1}{h} \sum_{m \leq h} \sum_{0 \leq |n-x| < m} f(n) - M_f(x, h)$ , with expected value  $M_f(x, h)$  (say,  $\approx h \sum_{d \leq x} g(d)/d$ ); so, this mean-square weights, on average, the  $f$ -values in (almost all, i.e. all, but  $o(N)$  possible exceptions) the short intervals  $[x - h, x + h]$ , with mild restrictions on  $h$  (say,  $h \rightarrow \infty$  and  $h = o(N)$ , when  $N \rightarrow \infty$ ).

## 1. Introduction and statement of the results.

We give upper bounds for the MODIFIED SELBERG INTEGRAL (for the Selberg integral [C-S], [C1], [C2], [C3])

$$\tilde{J}_f(N, h) \stackrel{\text{def}}{=} \sum_{x \sim N} \left| \sum_{|n-x| \leq h} \left(1 - \frac{|n-x|}{h}\right) f(n) - M_f(x, h) \right|^2 = \sum_{x \sim N} \left| \frac{1}{h} \sum_{m \leq h} \sum_{0 \leq |n-x| < m} f(n) - M_f(x, h) \right|^2,$$

(now on  $x \sim N$  is  $N < x \leq 2N$  in sums), where the MEAN-VALUE (averaged sum’s expected value) is  $(\forall \varepsilon > 0)$

$$M_f(x, h) \stackrel{\text{def}}{=} h \sum_{d \leq x+h} \frac{g(d)}{d} = h \sum_{d \leq x} \frac{g(d)}{d} + \mathcal{O}\left(\frac{h}{x} \sum_{x < d \leq x+h} |g(d)|\right) = h \sum_{d \leq x} \frac{g(d)}{d} + \mathcal{O}_\varepsilon\left(\frac{h^2 x^\varepsilon}{x}\right),$$

with  $g := f * \mu$  (see [T]), here for the class of ESSENTIALLY BOUNDED arithmetic real functions  $f$ . We use “ESSENTIALLY” to leave (as they’re negligible) arbitrarily small powers of  $N$ . With Vinogradov notation [D]

$$f \text{ is ESSENTIALLY BOUNDED (abbrev. } f \lll 1) \iff \forall \varepsilon > 0 \quad f(n) \ll_\varepsilon n^\varepsilon$$

(here,  $\forall n \leq 2N + h$ : we “don’t see”  $f$  any further; hence,  $f(n) \ll_\varepsilon N^\varepsilon$ ), while  $G$  essentially bounds  $F$  when

$$F(N, h) \lll G(N, h) \iff \forall \varepsilon > 0 \quad |F(N, h)| \ll_\varepsilon N^\varepsilon G(N, h).$$

As an application,  $([x])$  is the INTEGER PART of  $x \in \mathbb{R}$   $f \lll 1$  ( $\Leftrightarrow g \lll 1$ , by Möbius inversion [T]) gives

$$\begin{aligned} \int_N^{2N} \left| \sum_{|n-x| \leq h} \left(1 - \frac{|n-x|}{h}\right) f(n) - M_f(x, h) \right|^2 dx &\lll \int_N^{2N} \left| \sum_{|n-[x]| \leq h} \left(1 - \frac{|n-[x]|}{h}\right) f(n) - M_f([x], h) \right|^2 dx + N \\ &\lll \sum_{N \leq x < 2N} \left| \sum_{|n-x| \leq h} \left(1 - \frac{|n-x|}{h}\right) f(n) - M_f(x, h) \right|^2 + N \lll \tilde{J}_f(N, h) + h^2 + N. \end{aligned}$$

Hence, leaving  $\lll N + h^2$ , this integral (continuous mean-square) bound comes from the one for  $\tilde{J}_f(N, h)$ .

Our main result is the following. (We call the  $G * \mathbf{1}$  in the following a WINTNER MAJORANT of  $g * \mathbf{1}$ .)

**THEOREM.** *Let  $N, h, Q \in \mathbb{N}$ , with EVEN  $h \rightarrow \infty$ ,  $h = o(N)$  when  $N \rightarrow \infty$  and  $Q \leq N + h$ . Let  $g : \mathbb{N} \rightarrow \mathbb{R}$  have  $\text{supp}(g) \subset [1, Q]$ . Then*

$$|g| \ll G \lll 1 \text{ (IN THE SUPPORTS INTERSECTION)} \Rightarrow \tilde{J}_{g*\mathbf{1}}(N, h) \lll \tilde{J}_{G*\mathbf{1}}(N, h) + Nh.$$

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The same arguments in §3 prove the

PROPOSITION. Let  $N, h \in \mathbb{N}$ , with EVEN  $h \rightarrow \infty$  and  $h = o(N)$  when  $N \rightarrow \infty$ . Assume  $Q : \mathbb{R} \rightarrow \mathbb{R}$  is a strictly increasing function, with  $1 \leq Q(x) \leq x$ ,  $\forall x \geq 1$ . Let  $g$  be REAL AND (independent of  $N, h$  and  $x$ ) with support RESTRICTED TO, inside  $x$ -mean-square, the interval  $[1, Q(x+h)]$ ,  $\forall x \sim N$ . Then

$$|g| \ll G \ll 1 \text{ (IN THE SUPPORTS INTERSECTION)} \Rightarrow \tilde{J}_{g*1}(N, h) \ll \tilde{J}_{G*1}(N, h) + Nh.$$

We start proving the Theorem. (The proof will be completed in §3, where we'll prove the Proposition, too.)

PROOF. Assume now on  $Q \leq N + h$ ,  $g : \mathbb{N} \rightarrow \mathbb{R}$ , with  $\text{supp}(g) \subset [1, Q]$ ,  $g * 1 := f \ll 1$ ,  $h \rightarrow \infty$  and  $h = o(N)$ .

$$\tilde{J}_f(N, h) = \sum_{x \sim N} \left| \sum_{q \leq Q} g(q) \tilde{\chi}_q(x) \right|^2,$$

where we define, this time (compare [C-S], esp.),  $\forall q \in \mathbb{N}$ , (even if we need it  $\forall q \leq Q$ )

$$\tilde{\chi}_q(x) \stackrel{\text{def}}{=} \sum_{\substack{|n-x| \leq h \\ n \equiv 0 \pmod{q}}} \left(1 - \frac{|n-x|}{h}\right) - \frac{h}{q} = \sum_{\substack{\ell|q \\ \ell > 1}} \frac{\ell}{q} \sum_{j \leq \frac{\ell}{2}}^* \tilde{c}_{j,\ell} \cos \frac{2\pi x j}{\ell}, \text{ WHERE}$$

the FOURIER COEFFICIENTS ARE POSITIVE (better, non-negative), say from Fejér's kernel,

$$\tilde{c}_{j,q} := \frac{1}{q} \left( \frac{2}{h} \frac{\sin^2 \pi j h / q}{\sin^2 \pi j / q} \right) := \frac{1}{q} \tilde{F}_h \left( \frac{j}{q} \right) \geq 0 \quad \forall j \leq \frac{q}{2}$$

and (use Parseval identity [C-S]), writing henceforth  $\sum^*$  to SUM OVER REDUCED RESIDUE CLASSES,

$$\sum_{j \leq q}^* |\tilde{c}_{j,q}|^2 \ll \sum_{j \leq q} |\tilde{c}_{j,q}|^2 \ll \left\| \frac{h}{q} \right\| \ll \min \left( 1, \frac{h}{q} \right).$$

Here, as usual,  $\|\alpha\| := \min_{n \in \mathbb{Z}} |\alpha - n|$  is the DISTANCE TO the INTEGERS,  $\forall \alpha \in \mathbb{R}$ . (In fact,

$$\tilde{\chi}_q(x) = \frac{1}{q} \sum_{j \not\equiv 0 \pmod{q}} \sum_{|s| \leq h} (1 - |s|/h) e_q(js) e_q(xj),$$

from ORTHOGONALITY OF ADDITIVE CHARACTERS [V]; from  $h$  EVEN, use Fejér kernel summation,

$$\begin{aligned} \tilde{\chi}_q(x) &= \frac{1}{q} \sum_{0 < |j| \leq q/2} \left( \frac{1}{h} \frac{\sin^2 \pi j h / q}{\sin^2 \pi j / q} \right) \cos \frac{2\pi x j}{q} = \frac{1}{q} \sum_{j \leq q/2} \left( \frac{2}{h} \frac{\sin^2 \pi j h / q}{\sin^2 \pi j / q} \right) \cos \frac{2\pi x j}{q} = \\ &= \sum_{\substack{d|q \\ d < q}} \sum_{\substack{j \leq q/2 \\ (j,q)=d}} \tilde{c}_{j,q} \cos \frac{2\pi x j}{q} = \sum_{\substack{d|q \\ d < q}} \frac{1}{d} \sum_{\substack{j' \leq q/(2d) \\ (j', (q/d))=1}} \tilde{c}_{j',q/d} \cos \frac{2\pi x j'}{q/d} = \sum_{\substack{\ell|q \\ \ell > 1}} \frac{\ell}{q} \sum_{\substack{j \leq \ell/2 \\ (j,\ell)=1}} \tilde{c}_{j,\ell} \cos \frac{2\pi x j}{\ell}, \end{aligned}$$

where, as in [C-S], we use that  $\tilde{c}_{dj',dq'} = \tilde{c}_{j',q'}/d$ ,  $\forall d, j', q' \in \mathbb{N}$  and we "FLIP" the divisors  $\ell := q/d$ .)

THE RAMANUJAN COEFFICIENTS OF OUR  $f : \mathbb{N} \rightarrow \mathbb{C}$  ARE  $R_\ell(f) \stackrel{\text{def}}{=} \sum_{m \equiv 0 \pmod{\ell}} \frac{(f * \mu)(m)}{m} \quad \forall \ell \in \mathbb{N}$

(WELL-DEFINED, SINCE  $g := f * \mu$  HAS  $|\text{supp}(g)| < \infty$ ) TO GET (with:  $\text{supp}(g) \subset [1, Q]$  and  $g \ll 1$ )

$$(0) \quad |g| \ll G \ll 1 \Rightarrow R_\ell(g * 1) = \frac{1}{\ell} \sum_{q \leq \frac{Q}{\ell}} \frac{g(\ell q)}{q} \ll R_\ell(G * 1) \ll \frac{1}{\ell} \sum_{q \leq \frac{2N+h}{\ell}} \frac{1}{q} = R_\ell(1 * 1) := R_\ell(d) \ll \frac{1}{\ell}.$$

Here  $d(n) \stackrel{\text{def}}{=} \sum_{q|n} 1 \ll 1$  is the DIVISOR FUNCTION. We need the following Lemma.

## 2. An elementary Lemma.

This inequality, saying  $g \lll 1 \Rightarrow R_q(g * \mathbf{1}) \lll R_q(\mathbf{1} * \mathbf{1})$  ( $\forall q$ , here), is the core of the general philosophy underlying our THEOREM: we use a kind of “majorant”, for (real) essentially bounded functions  $f(n)$ , represented by the divisor function,  $d(n)$ . However, for the time being, we use  $|g| \ll G \lll 1$  in our bounds.

Also, THESE COEFFICIENTS ALLOW US, THEN, TO WRITE

$$\begin{aligned} \tilde{J}_f(N, h) = & \sum_{1 < \ell \leq Q} R_\ell^2(f) \sum_{j \leq \frac{\ell}{2}}^* \tilde{F}_h^2\left(\frac{j}{\ell}\right) \sum_{x \sim N} \cos^2 \frac{2\pi x j}{\ell} + \\ & + 2 \sum_{1 < \ell, t \leq Q} R_\ell(f) R_t(f) \sum_{\substack{j \leq \frac{\ell}{2} \\ \frac{j}{\ell} - \frac{r}{t} > 0}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos \frac{2\pi x j}{\ell} \cos \frac{2\pi x r}{t} \end{aligned}$$

(use above properties of  $\chi_q$  and (0) above), since for the FAREY FRACTIONS (they’re reduced ones, this time in  $[0, 1/2]$ )  $j/\ell = r/t \Rightarrow j = r, \ell = t$  and we may exchange the couples whenever  $j/\ell < r/t$ . Hence, say,

$$(1) \quad \tilde{J}_f(N, h) = \tilde{D}_f(N, h) + \sum_{1 < \ell, t \leq Q} R_\ell(f) R_t(f) \sum_{\substack{j \leq \frac{\ell}{2} \\ \frac{j}{\ell} - \frac{r}{t} > 0}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \left( \sum_{x \sim N} \cos 2\pi \delta x + \sum_{x \sim N} \cos 2\pi \sigma x \right),$$

where, say,

$$\tilde{D}_f(N, h) \stackrel{def}{=} \sum_{1 < \ell \leq Q} R_\ell^2(f) \sum_{j \leq \frac{\ell}{2}}^* \tilde{F}_h^2\left(\frac{j}{\ell}\right) \sum_{x \sim N} \cos^2 \frac{2\pi x j}{\ell} \geq 0$$

is the DIAGONAL and, say,  $\delta := \left\| \frac{j}{\ell} - \frac{r}{t} \right\| = \frac{j}{\ell} - \frac{r}{t} > 0$ ,  $\sigma := \left\| \frac{j}{\ell} + \frac{r}{t} \right\| \in [0, \frac{1}{2}]$ . Since  $\tilde{F}_h(j/q) \geq 0 \ \forall j \leq q/2$ ,

$$\begin{aligned} & \sum_{1 < \ell, t \leq Q} R_\ell(f) R_t(f) \sum_{\substack{j \leq \frac{\ell}{2} \\ r \leq \frac{t}{2} \\ 0 < \delta := \frac{j}{\ell} - \frac{r}{t} \leq \frac{1}{A}}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \delta x \lll \\ & \lll \sum_{1 < \ell, t \leq Q} R_\ell(G * \mathbf{1}) R_t(G * \mathbf{1}) \sum_{\substack{j \leq \frac{\ell}{2} \\ r \leq \frac{t}{2} \\ 0 < \delta := \frac{j}{\ell} - \frac{r}{t} \leq \frac{1}{A}}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \delta x, \end{aligned}$$

AND

$$\begin{aligned} & \sum_{1 < \ell, t \leq Q} R_\ell(f) R_t(f) \sum_{\substack{j \leq \frac{\ell}{2} \\ r \leq \frac{t}{2} \\ \delta > 0, \sigma := \left\| \frac{j}{\ell} + \frac{r}{t} \right\| \leq \frac{1}{A}}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \sigma x \lll \\ & \lll \sum_{1 < \ell, t \leq Q} R_\ell(G * \mathbf{1}) R_t(G * \mathbf{1}) \sum_{\substack{j \leq \frac{\ell}{2} \\ r \leq \frac{t}{2} \\ \delta > 0, \sigma := \left\| \frac{j}{\ell} + \frac{r}{t} \right\| \leq \frac{1}{A}}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \sigma x, \end{aligned}$$

PROVIDED  $N = o(A)$ , when  $N \rightarrow \infty$ . In fact, Taylor expansion of  $\cos$  gives in both cases a positive  $x$ -sum. Here we used estimates for Ramanujan coefficients coming from (0) bounds.

This “MAJORANT PRINCIPLE” is not applicable to all of our  $\tilde{J}_f(N, h)$ , as the terms for which  $\delta$  or  $\sigma$  are above  $1/A \rightarrow 0$ , say; better,  $1/A = o(1/N)$  here) are troublesome: we don’t know the sign of the  $x$ -sum.

However, luckily enough, we are able bound their contribution to our integral using a very simple “WELL-SPACED” argument, to be explicit the one used to prove Large Sieve type inequalities, applied to Farey fractions (as we have here, indeed). This has been done in Lemma 2 [C-S] (uses only Cauchy inequality); as in our present ELEMENTARY Lemma, following.

We can state and show our

LEMMA. Let  $N, h \in \mathbb{N}$  with  $h \rightarrow \infty$  and  $h = o(N)$  when  $N \rightarrow \infty$ . ASSUME  $g : \mathbb{N} \rightarrow \mathbb{C}$  with  $g(q) = 0$   $\forall q > Q$ , where  $1 \leq Q \ll N$ . Set  $f := g * \mathbf{1}$ . Choose  $A \in \mathbb{R}$ ,  $A = A(N, h) \rightarrow \infty$  when  $N \rightarrow \infty$ . THEN

$$\sum_{1 < \ell, t \leq Q} R_\ell(f) R_t(f) \sum_{\substack{j \leq \frac{\ell}{2} \\ \delta := \frac{j}{\ell} - \frac{r}{t} > 1/A}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \delta x \ll Ah,$$

$$\sum_{1 < \ell, t \leq Q} R_\ell(f) R_t(f) \sum_{\substack{j \leq \frac{\ell}{2} \\ \delta > 0, \sigma := \left\| \frac{j}{\ell} + \frac{r}{t} \right\| > 1/A}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \sigma x \ll Ah.$$

PROOF. The elementary calculation of the following exponential sum (compare [D, ch.25]) gives

$$\alpha \notin \mathbb{Z} \Rightarrow \sum_{x \sim N} e(\alpha x) \ll \frac{1}{\|\alpha\|},$$

which, together with (recall  $\tilde{c}_{j,q} := \tilde{F}_h(j/q)/q$ , here use (0) bounds, from  $f \ll 1$ )

$$\sum_{1 < \ell, t \leq Q} R_\ell(f) R_t(f) \sum_{\substack{j \leq \frac{\ell}{2} \\ \delta := \frac{j}{\ell} - \frac{r}{t} > 1/A}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \delta x \ll \sum_{1 < \ell, t \leq Q} \sum_{|j| \leq \frac{\ell}{2}}^* \sum_{|r| \leq \frac{t}{2}}^* \frac{|\tilde{c}_{j,\ell}| \cdot |\tilde{c}_{r,t}|}{\left\| \frac{j}{\ell} - \frac{r}{t} \right\|}$$

and, changing sign to  $r$ ,

$$\sum_{1 < \ell, t \leq Q} R_\ell(f) R_t(f) \sum_{\substack{j \leq \frac{\ell}{2} \\ \delta > 0, \sigma := \left\| \frac{j}{\ell} + \frac{r}{t} \right\| > 1/A}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \sigma x \ll \sum_{1 < \ell, t \leq Q} \sum_{|j| \leq \frac{\ell}{2}}^* \sum_{|r| \leq \frac{t}{2}}^* \frac{|\tilde{c}_{j,\ell}| \cdot |\tilde{c}_{r,t}|}{\left\| \frac{j}{\ell} - \frac{r}{t} \right\|},$$

GIVE  $\frac{1}{A}$  WELL-SPACED (FAREY) FRACTIONS (doesn't matter where, in  $[-\frac{1}{2}, \frac{1}{2}]$  or  $[0, 1]$  here); then,

$$\Sigma := \sum_{1 < \ell, t \leq Q} \sum_{|j| \leq \frac{\ell}{2}}^* \sum_{\substack{|r| \leq \frac{t}{2} \\ \left\| \frac{j}{\ell} - \frac{r}{t} \right\| > 1/A}}^* \frac{|\tilde{c}_{j,\ell}| \cdot |\tilde{c}_{r,t}|}{\left\| \frac{j}{\ell} - \frac{r}{t} \right\|} \ll \sum_{1 < \ell \leq Q} \sum_{j \leq \ell}^* |\tilde{c}_{j,\ell}|^2 \sum_{1 < t \leq Q} \sum_{r \leq t}^* \frac{1}{\left\| \frac{r}{t} - \frac{j}{\ell} \right\|},$$

using Cauchy inequality (& variables symmetry); number the  $\mathcal{O}(Q^2)$  Farey fractions  $\lambda_m := \frac{j}{\ell}$ ,  $\lambda_n := \frac{r}{t}$ ,

$$n \neq m \Rightarrow \|\lambda_n - \lambda_m\| \geq \frac{|n - m|}{A}$$

which gives, recalling from the above

$$\sum_{j \leq \ell}^* |\tilde{c}_{j,\ell}|^2 \ll \sum_{j \leq \ell} |\tilde{c}_{j,\ell}|^2 \ll \left\| \frac{h}{\ell} \right\| \ll \min \left( 1, \frac{h}{\ell} \right),$$

the required

$$\Sigma \ll A \sum_{1 < \ell \leq Q} \sum_{j \leq \ell}^* |\tilde{c}_{j,\ell}|^2 \ll Ah,$$

since (in the sequel, let  $L := \log N$ )

$$\sum_{1 < t \leq Q} \sum_{r \leq t}^* \frac{1}{\left\| \frac{r}{t} - \frac{j}{\ell} \right\|} = \sum_{\substack{n \neq m \\ \|\lambda_n - \lambda_m\| > 1/A}} \frac{1}{\|\lambda_n - \lambda_m\|} \ll A \sum_{n \neq m} \frac{1}{|n - m|} \ll A \sum_{1 \leq k \leq Q^2} \frac{1}{k} \ll AL \ll A. \quad \square$$

### 3. Completion of the Theorem proof. Proof of the Proposition. Remarks.

We complete the proof, RECALLING THE MAJORANT PRINCIPLE, see above, with  $f := g * \mathbf{1}$ , SAY,  $F := G * \mathbf{1}$ :

$$\begin{aligned}
(2) \quad & \tilde{D}_f(N, h) + \sum_{1 < \ell, t \leq Q} R_\ell(f) R_t(f) \sum_{\substack{j \leq \frac{\ell}{2} \\ 0 < \delta \leq \frac{1}{A}}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \delta x + \\
& + \sum_{1 < \ell, t \leq Q} R_\ell(f) R_t(f) \sum_{\substack{j \leq \frac{\ell}{2} \\ \delta > 0, \sigma := \left\| \frac{j}{\ell} + \frac{r}{t} \right\| \leq \frac{1}{A}}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \sigma x \lll \\
& \lll \tilde{D}_F(N, h) + \sum_{1 < \ell, t \leq Q} R_\ell(F) R_t(F) \sum_{\substack{j \leq \frac{\ell}{2} \\ 0 < \delta \leq \frac{1}{A}}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \delta x + \\
& + \sum_{1 < \ell, t \leq Q} R_\ell(F) R_t(F) \sum_{\substack{j \leq \frac{\ell}{2} \\ \delta > 0, \sigma := \left\| \frac{j}{\ell} + \frac{r}{t} \right\| \leq \frac{1}{A}}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \sum_{x \sim N} \cos 2\pi \sigma x,
\end{aligned}$$

SAY, FOR  $\frac{1}{A} = o\left(\frac{1}{N}\right)$ ; SINCE THE (WELL-SPACED FAREY FRACTIONS) LEMMA GIVES plus  $\lll Ah$ , BOTH for  $f$  and  $F$  (in place of  $f$ ), CHOOSE  $A = NL$  (or EVEN  $A = N \log \log N$ , here!) IN ORDER TO GET

$$\tilde{J}_f(N, h) \lll \tilde{J}_{G * \mathbf{1}}(N, h) + Nh. \quad \square$$

We prove, now, the Proposition. (Abbrev.  $X(q) := \text{THE INVERSE } Q^{-1}(q): Q(X(q)) = q, X(Q(x)) = x \forall q, x$ .)

PROOF. Instead of (1) we have (recall from (0) the  $R_\ell(f)$  definition), WITH  $Q := Q(2N + h)$ , that  $\tilde{J}_f(N, h)$  is

$$\begin{aligned}
& \sum_{x \sim N} \left| \sum_{q \leq Q(x+h)} g(q) \tilde{\chi}_q(x) \right|^2 = \sum_{1 < \ell \leq Q} \sum_{d_1, d_2 \leq \frac{Q}{\ell}} \frac{g(\ell d_1) g(\ell d_2)}{d_1 d_2} \sum_{j \leq \frac{\ell}{2}}^* \tilde{F}_h^2\left(\frac{j}{\ell}\right) \sum_{\substack{x \sim N \\ x \geq \tilde{X}(\ell d_1) - h \\ x \geq \tilde{X}(\ell d_2) - h}} \cos^2 \frac{2\pi x j}{\ell} + \\
& + \sum_{1 < \ell, t \leq Q} \sum_{d \leq \frac{Q}{\ell}} \sum_{q \leq \frac{Q}{t}} \frac{g(\ell d) g(t q)}{d q} \sum_{\substack{j \leq \frac{\ell}{2} \\ \delta := \frac{j}{\ell} - \frac{r}{t} > 0}}^* \sum_{r \leq \frac{t}{2}}^* \tilde{F}_h\left(\frac{j}{\ell}\right) \tilde{F}_h\left(\frac{r}{t}\right) \left( \sum_{\substack{x \sim N \\ x \geq \tilde{X}(\ell d) - h \\ x \geq \tilde{X}(t q) - h}} \cos 2\pi \delta x + \sum_{\substack{x \sim N \\ x \geq \tilde{X}(\ell d) - h \\ x \geq \tilde{X}(t q) - h}} \cos 2\pi \sigma x \right)
\end{aligned}$$

and (i) the “MAJORANT PART” (I.E., DIAGONAL & NEARBY, (2) ABOVE) IS UNCHANGED, SINCE TAYLOR EXPANSION still gives NON-NEGATIVE  $x$ -SUMS, while (ii) the “WELL-SPACED PART”, i.e., THE LEMMA, STILL HOLDS, SINCE IN OUR LEMMA the initial estimate (of EXPONENTIAL SUMS, FROM [D], quoted) DOES NOT DEPEND ON THE  $x$ -INTERVAL (THOUGH MAYBE NON-OPTIMAL, say the length is  $o(1/\|\alpha\|)$ , esp.).  $\square$

We give some remarks, to have an idea of the (possible) applications of the Theorem (and/or the Proposition).

First of all, we may think to substitute  $G$  with a constant (say,  $G = \mathbf{1}$ ), but then the resulting modified Selberg integral of the divisor function does not have a non-trivial estimate, since we now force the expected value  $M_f(x, h)/h$  of  $f$  in the short interval to be of the kind  $\sum_{d \leq x} \frac{1}{d}$ , which is different from the actual one (roughly  $\log x + 2\gamma$ ,  $\gamma := \text{EULER-MASCHERONI CONSTANT}$ , instead of the present  $\log x + \gamma$ ); this results in a “trivial” modified Selberg integral. In fact, if we look (say) for example, at [C-S] Theorem 2 (even if about Selberg integral of the divisor function, not the modified one!), the expected value is calculated (in full agreement with classic residue-calculated terms, see [C4], for example) starting from  $g = \mathbf{1}$ , but after flipping the divisors (so, the range is not up to about  $x$ , but cut about  $\sqrt{x}$ !).

Second, the search for good (i.e., with non-trivial modified Selberg integral  $\tilde{J}_{G*1}(N, h)$ , say we gain small powers on the rough  $\mathcal{O}(Nh^2)$ -bound) majorants  $G$ , i.e. good Wintner majorants  $F := G * \mathbf{1}$  for the (real, essentially bounded)  $f := g * \mathbf{1}$  (i.e.,  $|g| \ll G$ ), is not a trivial question !

Third, it's not yet completely clear that this can (and how) give a "smoothing" of the arithmetic behind the function  $f := g * \mathbf{1}$  (substantially and morally, we should like to bound  $g$  with constants, but see the first consideration above !); compare (but there we make further hypotheses on the function  $f$ ) the appearance of the idea of majorant principles in our paper [C5] (where the attention is on non-negative exponential sums in the long range, not in the short, that's for free here !).

Last but not least, why did we study the modified Selberg integral and not simply the Selberg integral with this approach ? Simply because it fails, due to Fourier coefficients of non-constant sign coming from the short interval (see, instead, the coefficients  $\tilde{F}_h \geq 0$  above) ! What about the symmetry integral, then ...

### References

- [C1] Coppola, G.- *On the Correlations, Selberg integral and symmetry of sieve functions in short intervals* - <http://arxiv.org/abs/0709.3648v3> (to appear on: Journal of Combinatorics and Number Theory)
- [C2] Coppola, G.- *On the Correlations, Selberg integral and symmetry of sieve functions in short intervals, II* - Int. J. Pure Appl. Math. **58.3**(2010), 281–298.
- [C3] Coppola, G.- *On the Correlations, Selberg integral and symmetry of sieve functions in short intervals, III* - <http://arxiv.org/abs/1003.0302v1>
- [C4] Coppola, G.- *On the Selberg integral of the  $k$ -divisor function and the  $2k$ -th moment of the Riemann zeta-function* - <http://arxiv.org/abs/0907.5561v1> - to appear on Publ. Inst. Math., Nouv. Sr.
- [C5] Coppola, G.- *On the symmetry of arithmetical functions in almost all short intervals, V* - (electronic) <http://arxiv.org/abs/0901.4738v2>
- [C-S] Coppola, G. and Salerno, S.- *On the symmetry of the divisor function in almost all short intervals* - Acta Arith. **113** (2004), **no.2**, 189–201. [MR 2005a:11144](#)
- [D] Davenport, H.- *Multiplicative Number Theory* - Third Edition, GTM 74, Springer, New York, 2000. [MR 2001f:11001](#)
- [T] Tenenbaum, G.- *Introduction to Analytic and Probabilistic Number Theory* - Cambridge Studies in Advanced Mathematics, **46**, Cambridge University Press, 1995. [MR 97e:11005b](#)
- [V] Vinogradov, I.M.- *The Method of Trigonometrical Sums in the Theory of Numbers* - Interscience Publishers LTD, London, 1954. [MR 15,941b](#)

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